

Mass Spectrum and Number of Light Neutrinos: An Attempt of the Gauge Explanation

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Symplectic flavour symmetry group $Sp(n/2)$ (n is even) of n Majorana states does not allow for invariant Majorana masses. Only specific mass matrices with diagonal and nondiagonal elements are possible here. As a result of the spontaneous violation of flavour and chiral symmetries, a mass matrix could appear only for the number of flavours $n = 6$ and only together with R, L - symmetry violation (i.e., parity violation). The see-saw mechanism produces here three light and three heavy Dirac particles (neutrinos). The peculiarity of the observed light neutrino spectrum two states located far from the third one can be explained by certain simple properties of mass matrices appearing in $Sp(3)$. The ordering of the states corresponds to normal mass hierarchy. Situation, when neutrino mass differences are significantly less than masses themselves, appears to be unrealizable here. Mixing angles for neutrinos can not be determined without understanding formation mechanisms for charged lepton spectrum and Majorana state weak currents.

I. INTRODUCTION

Neutrinos have a mass. The spectrum and mixing of neutrinos are based on other principles than the respective properties of charged, and therefore compulsorily Dirac, massive states [1]. That is why, the thought that the Majorana properties (which only neutrinos may have) play a crucial part in the origin and character of neutrino spectrum is gaining wider support [2].

If so, the mechanism of neutrino mass generation has little in common with the mechanism of charged Dirac mass generation. Mixing angles are the simultaneous consequence of both mechanisms and therefore can not provide neutrino- specific information on the mass generation model. There are only the smallness of the neutrino mass as opposed to any other particles, the number of light neutrinos $N_\nu = 3$, and the large ratio of mass squared

differences for these light neutrinos (1) that indeed can be considered as characteristics of the neutrino mechanism. Using standard conventions [1], one has

$$23 \lesssim \frac{\Delta m_{23}^2}{\Delta m_{12}^2} \lesssim 40 \quad (1)$$

Relation (1) indicates that one of the states, "3", is located rather far from the other two, which are very close to each other. One distinguishes here direct and inverted hierarchies [2, 3].

The smallness of neutrino masses is the only aspect to this phenomenon that has a fine consistent explanation. The see-saw mechanism, developed and investigated by many authors (see review [3]), considers the smallness of neutrino masses as a result of the existence of a very high energy scale. All proposed models appear with participation of scalar interactions and scalar (Higgs) particles; they do not explain relation (1), nor do they distinguish between the types of hierarchy.

At the same time, all observed interactions are based on the exchange of gauge bosons (with a vector or tensor spin). The question is whether mass formation phenomena could similarly be attributed only to local gauge mechanisms. This, of course, would imply a dynamical way for mass formation, and result in inaccessible nonperturbative solutions. What only can be achieved really by this approach are direct symmetric consequences, and they are the subject of this paper.

Let us consider n Majorana states. For the flavor nonabelian gauge group of symplectic transformations $Sp(n/2)$, $n = 2, 4, 6, \dots$, the invariant Majorana masses, both for chiral right R and left L particles, are identically equal to zero. Thus, it is only the whole mass matrix, with both diagonal and nondiagonal elements, that could be produced here by dynamical spontaneous breaking. On the other hand, the Dirac part of the complete (R, L) matrix could be invariant.

The principal accomplishment of the present approach is equations (26)-(28) in Section 5. They provide a choice of conditions under which the spontaneous appearance of the mass matrix becomes possible in $Sp(n/2)$. There are two possible solutions: one is R, L -symmetry at $n = 2$, $Sp(1)$, and the other, the spontaneous breaking of R, L -symmetry at $n = 6$, $Sp(3)$. All neutrinos necessarily appear as Dirac ones. In the physical sense, the second solution, $n = 6$, is more interesting. It creates conditions under which the see-saw mechanism divides the six Dirac neutrinos into three light and three heavy ones. Note that these

conditions can only be realized for Majorana states: only then equations for spontaneous mass matrices become self-consistent, although their solutions resolve into exclusively Dirac massive particles.

Majorana mass distributions in $Sp(3)$ and the action of the see-saw mechanism permit such a disposition for light particles where one state is located far from the other two. Such a spectrum can be explained by a rather usual distribution of the roots of the cubic characteristic equation for Majorana mass squared which results in $Sp(3)$. The quantities of the system do not require any fine tuning, besides providing the see-saw situation. The light neutrino mass hierarchy is normal.

In Section 2, we discuss reasons for selecting $Sp(n/2)$ as a flavour symmetry in the Majorana problem. Section 3 investigates properties of mass matrices which are acceptable in $Sp(n/2)$. Section 4 proposes a gauge model considered for a hypothesis of spontaneous generation of mass matrices. It describes the properties that would allow dynamical violation of flavour and chiral symmetries. Section 5 discusses the conditions under which the proposed mechanism would work. Dirac states corresponding to the Majorana spectrum $Sp(3)$ are constructed in Section 6. Section 7 considers possible explanations in $Sp(3)$ for the observed light neutrino spectrum. Section 8, Conclusions, describes difficulties in transition to a realistic model, which includes charged leptons, i.e., the whole set of weak processes.

II. CHOOSING GAUGE GROUP

Majorana state flavours cannot transform under any representation of their symmetry group G_F . Indeed, the identity of the particle and antiparticle, which relates spinors $\Psi_a(x)$ and $C\bar{\Psi}^{Ta}(x)$ (where a is a flavour index, $a = 1, 2, \dots, n$), requires that conjugate (contragredient [4]) representations be equivalent. Conjugate representations are related by root reflections at the coordinate origin and by changes in infinitesimal operators: $t^A \rightarrow -t^{AT}$ ($t^{A+} = t^A$) [4]. For equivalent conjugate representations, there exists a matrix $n \times n$, h , that allows obtaining $-t^{AT}$ from t^A :

$$h^+ t^A h = -t^{AT}. \quad (2)$$

This matrix raises and lowers Ψ indices, and its properties and notations are as follows:

$$h = \{h_{ab}\}, \quad h^+ = \{h^{ab}\}, \quad hh^+ = 1, \quad h^{ab} = \pm h_{ab}, \quad h^T = \pm h. \quad (3)$$

The designation of h - is to renumber roots. The right diagonal of the matrix h contains exclusively ± 1 elements.

Let us first consider n massless Majorana states. In terms of common chiral operators $\psi_{RL} = 1/2(1 \pm \gamma_5)\psi$, there are two operators covariant with respect to G_F group which can be called "Majorana-like". In a four-component form, they are:

$$\Psi_{(R,L)a}(x) = \psi_{(R,L)a}(x) + (1, \gamma_5)h_{ab}C\bar{\psi}_{(R,L)}^{Tb}(x). \quad (4)$$

The factor $(1, \gamma_5)$ is associated with the sign of $h^T = \pm h$ in Formula (3): at $h^T = h$ one should take the unity matrix, and at $h^T = -h$, the matrix γ_5 . Normalization in (4) is chosen so that the energy of free massless Majorana particles can be expressed in a usual way for neutral states:

$$\bar{\psi}^a \hat{p} \psi_a = \frac{1}{2} \bar{\Psi}^a \hat{p} \Psi_a. \quad (5)$$

States (4) transform under the same representation of the flavour group G_F as $\psi_{(R,L)a}(x)$ and demonstrate "Majorana" properties:

$$\Psi_{(R,L)}(x) = (1, \gamma_5)hC\bar{\Psi}_{(R,L)}^T(x), \quad (6)$$

$$C^+ = C^T = -C.$$

In Eq. (6), one observes a complete analogy with the extended charge parity (G -parity) [6]: the transition to the antiparticle takes place simultaneously with the group operation. The appearance of γ_5 at $h^T = -h$ has significance only for massive states ($m \rightarrow -m$), and ultimately leads to important physical consequences (see sections 3,6,7).

Complex conjugate representations are equivalent in symmetric representations (with root reflection). These are primarily adjoint representations of all groups. There also exist fundamental representations which demonstrate the same property, i.e., representations of orthogonal $O(n)$, symplectic $Sp(n/2)$ (using designations of [5]), and exceptional groups [4, 5]. The n -dimensional representation $Sp(n/2)$, $n = 2, 4, 6, \dots$, has an antisymmetric h :

$$h^T = -h, \quad h^+ = -h. \quad (7)$$

All other cases are symmetrical matrices h .

Of greatest interest in (7) is that the dynamical violation of flavour and chiral symmetries gives rise here to a whole Majorana mass matrix. Indeed, only a matrix with non-zero

diagonal and nondiagonal elements can result in this case because the invariant Majorana mass here is equal to zero:

$$\bar{\Psi}_R^a(x)\Psi_{Ra}(x) = -\bar{\psi}_R(x)hC\bar{\psi}_R^T(x) - \psi_R^T(x)h^+C\psi_R(x) = 0, \quad (8)$$

which is similar also for Ψ_L . Identity (8) results from the anticommutation of operators ψ_R and matrix h and C properties. This identity implies that, under condition (7), the appearance of Majorana masses is only possible with the full breaking of flavour symmetry, to the point where a mass matrix without residual symmetries is created. This matrix will immediately presented states with various mass values in the spectrum.

On the other hand the Dirac mass can exist in $Sp(n/2)$ in invariant form

$$\bar{\Psi}_R^a(x)\Psi_{La}(x) = \bar{\psi}_R^a(x)\psi_{La}(x) - \bar{\psi}_L^a(x)\psi_{Ra}(x) \neq 0^1. \quad (9)$$

Under symmetric representations $h = h^T$, Majorana and Dirac masses may both be present even without symmetry breaking. This would be the simplest way for spontaneous violation to occur, with the result being equal masses for all flavours rather than a mass matrix.

We, therefore, shall consider those possibilities that may present, for neutrino masses, a hypothetic spontaneous violation of the flavour symmetry $Sp(n/2)$ under the fundamental representation n . Note that this symmetry of leptons can also be the local gauge symmetry of interaction with the vector field F_μ^A , $A = 1, 2, \dots, n(n+1)/2$. Inclusion of this interaction does not result in new anomalies, even if weak interactions are taken into account. Full symmetry breaking leads to the absence of Goldstone particles and to the formation of heavy masses simultaneously for all F_μ^A . Let us first consider the properties of an acceptable mass matrix in the gauge-invariant scheme.

III. MAJORANA MASS MATRIX PROPERTIES

In essence, dynamical flavour symmetry violation is the generation of vacuum averages of $\bar{\Psi}^a(x)\Psi_b(x)$ combinations for R - and L -operators. According to (8), the trace of such matrix will be equal to zero for RR - and LL - systems; however, individual, or all, matrix

¹ The form (7) corresponds to the imaginary Dirac mass. Transition to real mass values occurs after redetermination of $\Psi_{(R,L)}$ in Eq.(4): $\Psi \rightarrow (i, \gamma_5)\Psi$ (sect. 3).

elements will be other than zero. For RR - and LL -systems, we have:

$$(M_{RR})_a{}^b = \langle \bar{\Psi}_R^b(x) \Psi_{Ra}(x) \rangle, \quad (M_{LL})_a{}^b = \langle \bar{\Psi}_L^b(x) \Psi_{La}(x) \rangle. \quad (10)$$

Let us consider a symmetrical matrix M_a^b which is real, as equations for spontaneous violation parameters ("gap" equations for mass quantities [7]) will by all means be real (with CP conservation).

Majorana conditions (6) for the case under consideration are:

$$\Psi_R(x) = \gamma_5 h C \bar{\Psi}_R^T(x), \quad (11)$$

For the L operator, we have:

$$\Psi_L(x) = -\gamma_5 h C \bar{\Psi}_L^T(x). \quad (12)$$

Form (12) depends on the phase selected, which, for the given form (11), makes the Dirac mass (9) also real. Equations (11) and (12), and the properties of anticommutation Ψ result in the following, non-covariant form of the relation between the matrix elements M_a^b (for RR and LL):

$$M_a{}^b = -h_{aa'} M_{a'}{}^{b'} h^{b'b}. \quad (13)$$

The covariant form of this relation is $M^{+T} = -h^+ M h$. The matrix h elements are:

$$h_{ab} = -h^{ab} = (-1)^a \delta_{n+1-a,b}, \quad a, b = 1, 2, \dots, n. \quad (14)$$

Trace of the matrix M evidently vanishes (13). As well, condition (13) implies vanishing of sums of all principal minors of odd order for matrices (10). To prove it, it is sufficient to write the sought sum in the following form:

$$\frac{1}{n_M!} \varepsilon^{ab\dots a_1 a_2 \dots} M_{a_1}{}^{b_1} M_{a_2}{}^{b_2} \dots \varepsilon_{ab\dots b_1 b_2 \dots},$$

where the letters without indices, $ab\dots$, are rows (columns) that are not used in the calculation of the minors under consideration, and to apply relation (13) and the formula:

$$\varepsilon^{a_1 a_2 \dots} h_{a_1 a'_1} h_{a_2 a'_2} \dots = \varepsilon_{a'_1 a'_2 \dots}.$$

The eigenvalue equation for matrix (10) will contain only even powers of eigenvalues M_f . For each of the matrices M_{RR} and M_{LL} , there are $n/2$ states which differ in the value of M_f^2 . Thus, since a pair of Majorana states with equal masses (the mass sign does not

matter $\Psi \rightarrow \gamma_5 \Psi$ could be redefined) are equivalent to one four-component Dirac state, n is a general number of possible physical spinors (see sections 6 and 7).

Dirac form (9) admits the existence of the invariant mass μ_{RL} :

$$\mu_{RL_a}{}^b = \frac{1}{n} \left\langle \bar{\Psi}_R^c(x) \Psi_{Lc}(x) \right\rangle \delta_a{}^b. \quad (15)$$

One, therefore, may expect here that spontaneous breaking will try to cause the least possible symmetry destruction and the diagonal matrix (15) will be a solution to the equations for the symmetry violation parameters ("gap" equations). In addition, formula (15) is necessary for our problem as it presents one of the conditions (Section 5) that make the existence of "gap" equations possible. The remainder of this section is devoted to the explanation of this statement.

Properties (11) and (12) result in the following equation for the product of the operators $\bar{\Psi}_R^a \Psi_{Lb}$:

$$\bar{\Psi}_R^a(x) \Psi_{Lb}(x) = h_{bb'} \bar{\Psi}_L^{b'}(x) \Psi_{Ra'}(x) h^{a'a}. \quad (16)$$

For real vacuum averages of these quantities, the elements of $\mu_{RL_a}{}^b$ become related:

$$(\mu_{RL})_a{}^b = h_{bb'} (\mu_{RL})_{a'}{}^{b'} h^{a'a}, \quad (17)$$

where μ_{RL} is an arbitrary real matrix that is related to the matrix μ_{LR} as follows: $\mu_{RL_a}{}^b \equiv \mu_{LR_b}{}^a$. For h from (14), Eq. (17) is automatically satisfied by the diagonal form (16) but leads to $n^2/2$ ratios for arbitrary forms of μ_{RL} matrices.

An additional $n^2/4$ relations for each of the symmetrical matrices M_{RR} and M_{LL} result from Eq.(13). One would argue that this may prevent appearance of a symmetrical $(2n \times 2n)$ matrix with the properties under discussion from any system of "gap" equations in the problem of spontaneous breaking:

$$M = \begin{vmatrix} M_{RR} & \mu_{RL} \\ \mu_{LR} & M_{LL} \end{vmatrix} \quad (18)$$

Indeed, "gap" equations should be formulated for each of the matrix M elements. Matrices M_{RR} and M_{LL} are general matrices with non-diagonal elements. The system, then, consists of $n(2n + 1)$ equations for the matrix M elements, supplemented with n^2 relations (13), (17). The number of variables in this system is $n(2n + 1)$: $n(2n - 1)$ free parameters of the orthogonal matrix diagonalizing M and $2n$ of its eigenvalues. The number of variables is less than the number of equations.

In the critical problem, a solution exists upon reaching a certain "critical" value of the effective interaction force characteristics (such as the coupling constants in the Nambu-Jona-Lasinio model, NJL [7]); at that, the critical parameters should be determined unambiguously. The number of governing equations must be equal to the number of parameters. One therefore has to look for a way to change the ratio between these two numbers.

This can be achieved if there exists some symmetry of interactions forming "gap" equations, in which case some of Ψ , as well as $V\Psi$, appear to be solutions to the system. Of interest for a real set of equations are real groups only. For the critical parameters to be defined unambiguously, the overall number of equations (including those noninvariant with respect to V) may exceed the number of unknown quantities by the number of free parameters V . Another option is: The set of basic equations results in solutions for which additional relations are fulfilled automatically; this, for example, happens if part of corresponding elements vanishes. A similar situation takes place if the diagonal form (15) is the only possible solution for the μ_{RL} part of the whole matrix M . At the same time, the matrices M_{RR} and M_{LL} can take neither the invariant form of Majorana masses nor the diagonal form (see the last portion of Section 4), and "gap" equations need to be formulated for each of the matrix M elements.

In all cases when the set of equations may have a solution, one should expect that the dynamics themselves will bring the system to required values, because the region under consideration will correspond to the energy minimum. In the sections that follow, we will see that the requirement of solution unambiguity imposes rigid conditions on the choice of the system with the symmetry $Sp(n/2)$ in which the proposed mechanism is able to work.

IV. GAUGE-INVARIANT MODEL FOR MAJORANA FLAVOURS

The local gauge interaction with the vector meson $F_\mu^A(x)$ seems to be the most preferable way for incorporating $Sp(n/2)$ into the problem of Majorana masses. For purely vector interactions of massless fermions, in particular, mass creation can only be associated with the dynamic violation of chiral (and flavour) symmetry, i.e., with the generation of vacuum averages (10) and (15).

Currents that define the vector interaction $F_\mu^A(x)$ with chiral fermions $\psi_{(R,L)}^a(x)$:

$$j_{(R,L)\mu}^A(x) = \bar{\psi}_{(R,L)}^a(x) \gamma_\mu t_a^{Ab} \psi_{(R,L)b}(x) \quad (19)$$

are directly rewritten with the Majorana operators $\Psi_{(R,L)}(x)$ (4). From formulae (4), (2), and anticommutation $\psi_{(R,L)}(x)$ we obtain:

$$J_{(R,L)\mu}^A(x) = \bar{\Psi}_{(R,L)}^a(x) \gamma_\mu t_a^{Ab} \Psi_{(R,L)b}(x) \equiv 2j_{(R,L)\mu}^A(x). \quad (20)$$

Note that the axial current of the Majorana states (4) with matrices having property (2) identically vanishes:

$$J_\mu^{(5)A}(x) = \bar{\Psi}(x) \gamma_\mu \gamma_5 t^A \Psi(x) = 0 \quad (21)$$

for the R - and L -systems.

For currents with matrices symmetrical with respect to (3) (unity matrix or matrix σ^P , see Appendix 2), we observe the opposite situation: vector currents formed with Majorana operators are equal to zero, whereas axial currents are equal to chiral vector currents.

Direct solution of the dynamic spontaneous breaking problem is not attainable under such a system. For less complex fermion models, solving this problem has been attempted numerous times both analytically [8] and by means of lattice computations [9].

We are interested in the symmetry properties of interactions between Majorana particles. Dependent on these properties is the unambiguity of the solution of equations for spontaneous violation parameters, i.e., "gap" equations. Two mechanisms are simultaneously engaged in the problem: vector particle F_μ^A mass generation and fermion mass generation. In NJL models [7], these two mechanisms, most likely interrelated, are represented by various combinations of Feinman diagrams.

The effective potential between Majorana fermions of interest to us would be easy to determine if integration over field $F_\mu^A(x)$ could be fulfilled in the functional integral for amplitudes. For R - and L -operators used as Ψ and $\bar{\Psi}$, the solution will depend on combinations (local and unlocal) of the following type:

$$\bar{\Psi}^a Z_1 \Psi_a, \bar{\Psi}^a Z_2 \Psi_b \bar{\Psi}^b Z_3 \Psi_a, \dots$$

Operators Z do not contain indices $Sp(n/2)$. At that, quantities such as $h^{ab} \Psi_a^T \dots \Psi_b$ will be transformed to $\bar{\Psi}^a \dots \Psi_a$ using Majorana formulae (11, 12), and products $t_a^A b t_c^A d$ will bring us to the same result if we use the formulae given below (23) or in Appendix 2 (A.8 and A.9). This circumstance is responsible for the difference between investigations using chiral and Majorana operators. Up to this point, the two notations were in a form of simple

variable change. In Majorana terms, additional real symmetry is achieved which can make possible the solution for "gap" equations of spontaneous breaking (Section 5).

These equations are, therefore, formed by interaction with the real, globally invariant group $O_L(n) = O_R(n) \equiv O(n)$ with $n(n-1)/2$ arbitrary parameters.

To make it more clear, let us consider V_{eff} in the second order, with the mechanism of mass generation F_μ^A isolated, by introducing an auxiliary scalar field with non-zero vacuum averages, as described in Appendix 1. If the mass M_F is heavy, the effective interaction at energies much lower than M_F will be the coupling "current \times current" (for $R \times R$ and $L \times L$):

$$V_{eff} = -\frac{g_F^2}{2M_F^2} j_\mu^A(x) j^{A\mu}(x) = -\frac{g_F^2}{8M_F^2} J_\mu^A(x) J^{A\mu}(x). \quad (22)$$

This formula can be identically transformed by means of relations (such as the Firz relations) for $Sp(n/2)$. We obtain:

$$\sum_A t_a^{Ab} t_c^{Ad} = \frac{1}{4} (\delta_a^d \delta_c^b - h_{ca} h^{bd}). \quad (23)$$

Eq. (23) is worked out in Appendix 2 (Formula A8).

Using (23) and the Firz identities for the $\gamma \times \gamma$ product between spinors RR , LL , and RL , we obtain (omitting arguments x in the operators):

$$\begin{aligned} V_{RR} &= \frac{g_F^2}{4M_F^2} \left(\bar{\Psi}_R^a \frac{1-\gamma_5}{2} \Psi_{Rb} \right) \left(\bar{\Psi}_R^b \frac{1+\gamma_5}{2} \Psi_{Ra} \right) = \\ &= \frac{g_F^2}{16M_F^2} \left[\left(\bar{\Psi}_R^a \Psi_{Rb} \right) \left(\bar{\Psi}_R^b \Psi_{Ra} \right) - \left(\bar{\Psi}_R^a \gamma_5 \Psi_{Rb} \right) \left(\bar{\Psi}_R^b \gamma_5 \Psi_{Ra} \right) \right] \end{aligned} \quad (24)$$

A similar result is obtained for $V_{LL} : \Psi_R \rightarrow \Psi_L$. Transforming (22) into (24) takes into account equations (7) and (21). For the product of currents $R \times L$, we have:

$$\begin{aligned} V_{RL} &= -2 \frac{g_F^2}{4M_F^2} \left[\left(\bar{\Psi}_R^a \frac{1-\gamma_5}{2} \Psi_{La} \right) \left(\bar{\Psi}_L^b \frac{1+\gamma_5}{2} \Psi_{Rb} \right) - \right. \\ &\quad \left. - \left(\bar{\Psi}_R^a \frac{1-\gamma_5}{2} \Psi_{Lb} \right) \left(\bar{\Psi}_L^b \frac{1+\gamma_5}{2} \Psi_{La} \right) \right]. \end{aligned} \quad (25)$$

Majorana conditions (11,12) permit us to prove symmetry of Eq.(25) relative to the transposition $R \leftrightarrow L$.

In a single-flavour system, the difference of the signs in formulae (24) and (25) would have an important physical meaning: repulsion in the particle-particle (antiparticle- antiparticle) and attraction in the particle-antiparticle system. In the multi-flavour nonabelian problem, this meaning is lost; however, it provides evidence against the appearance of diagonal RR - and LL - mass matrices in the system under consideration (Abelian version).

The above physical argument also indicates that in the gauge flavour symmetry scheme there probably is not such a variant of the neutrino spectrum where neutrino mass differences are much less than masses themselves. That situation may take place when all three roots of the eigenvalue equation for the Majorana mass matrix are approximately equal (see Section 7), which is only possible when this matrix has a near-diagonal form (Eq. (43)).

V. RECONCILIATION OF CONDITIONS FOR MATRIX M SPONTANEOUS APPEARANCE

The set of "gap" equations uses independent elements of the orthogonal matrix diagonalizing (18), and eigenvalues M , i.e., Majorana masses, as sought critical parameters. Our real mass matrix problem is limited to only real transformations.

All factors used, including particle propagators in Feynman graphs (if spontaneous violation equations are built similarly to NJL models [7]), are expressed through these unknown quantities. Overall, there are $n(2n + 1)$ equations and $n(2n + 1)$ parameters.

On the other hand, there also are relations (13), (17) imposed on the matrix elements by the "Majorana-ness" of Ψ_R , and Ψ_L . These "spare" equations need to be compensated for by tuning conditions as described at the end of Section 3. There are a few options available here.

Firstly, interactions between Majorana fermions are invariant with respect to orthogonal transformations $O(n) = O_R(n) = O_L(n)$, so that the total number of equations may exceed the number of variables by $n(n - 1)/2$, i.e., the number of the independent elements of the $O(n)$ group.

Secondly, spontaneous breaking creates an invariant form of the matrix μ_{RL} , or the diagonal matrix (15). The system will try to cause the least possible symmetry breaking. Under Majorana-ness conditions (11), (12), the matrix is symmetric: $\mu_{RL} = \mu_{LR}$. In this case, therefore, a part of $n^2/2$ conditions (17) repeat the part of $n(n + 1)/2$ equations for elements of symmetrical matrix with equal diagonal terms. It is these $n(n + 1)/2$ equations that should be included in the whole set.

Thirdly, there are two variants that should be considered for M_{RR} and M_{LL} :

a) R, L -symmetry. Spontaneous creation of both Majorana mass matrices M_{RR} and M_{LL} , with the matrices being identical and R, L -symmetry, hence parity, maintained.

b) R, L -asymmetry. Let us assume that $M_{LL} = 0$ (or $M_{RR} = 0$)².

This condition is not a result of solution. Therefore, the number of equations for M_{LL} that should be considered in this variant is $n(n+1)$: $n(n+1)/2$ conditions for the elements of the symmetric matrix $M_{LL_a}^b = 0$ and $n(n+1)/2$ "gap" equations for these elements from the general system for M (at that, $n^2/4$ additional conditions (13) are automatically fulfilled). Spontaneous violation of both R, L - symmetry and parity takes place.

Let us write out the resultant relationships: the number of equations minus the number of variables is equal to the number of independent symmetry parameters of the equations. We have:

$$\frac{n(n+1)}{2} + \frac{n^2}{4} + \left\{ \frac{\frac{n(n+1)}{2} + \frac{n^2}{4}}{n(n+1)} \right\} + \frac{n(n+1)}{2} - n(2n+1) = \frac{n(n-1)}{2} \quad (26)$$

for both the (a) and (b) variants. For the symmetry variant (a), where $M_{RR} = M_{LL}$, we obtain ($n \neq 0$):

$$n = 2. \quad (27)$$

For asymmetry variant (b), where $M_{RR} \neq 0$, $M_{LL} = 0$, the solution is

$$n = 6. \quad (28)$$

Other variants of choosing conditions do not give any physically plausible results.

It is worth being mentioned again (see Section 3) that if the "gap" equations do have a solution, this solution should describe a certain energy minimum, and the dynamics themselves will guide the system into the right region of parameter space.

The solution $n = 6$, Eq.(28), is of particular interest. The equation $M_{LL} = 0$ is a necessary condition for the see-saw mechanism to step in (see Section 6). We should also remember that the Majorana matrix M_{RR} has pairwise similar (in module) eigenvalues. Then, if the scale of M masses for M_{RR} is much higher than the Dirac μ , $n = 6$ means that the scheme contains three Dirac neutrinos of tiny mass, $\sim \mu^2/M$, and three of huge mass, $\sim M$. We will clarify these statements in Section 6, while concluding this section with the following remark.

² We choose $M_{LL} = 0$. Such a choice corresponds to usual consideration of the see-saw mechanism and seems to facilitate weak interaction insertion.

Destruction of the $O(n)$ group (as a result of fixing its parameters while solving the equations) will not lead to the generation of new Goldstone states. The $O(n)$ group is the symmetry group of the low-energy V_{eff} , and within the framework of the whole problem it represents a part of the fully broken $Sp(n/2)$ group. Consequently, Goldstone states $O(n)$ must have already been absorbed by the formation of heavy masses for F_μ^A . In the variant (b) R, L -symmetry violation, a massless Goldstone state is also absent. Here, similarly to the $U(1)$ -problem of QCD cite10, one observes a neutral chiral anomaly with F_μ^A -particles. A possible Goldstone particle will be massive.

VI. DIAGONALIZATION OF THE MATRIX M AND TRANSITION TO DIRAC STATES

Let us consider the variant $n = 6$, $M_{LL} = 0$, the most interesting from the physical point of view. We assume that the Majorana mass scale is much larger than the Dirac mass scale:

$$M \gg \mu. \quad (29)$$

In this case, the matrix M , Eq. (18), is easy to diagonalize in two steps. At first, we diagonalize M_{RR} by the orthogonal matrix U , i.e., with the transformation:

$$\Psi'_R = U\Psi_R. \quad (30)$$

Simultaneously, a transition to $\Psi'_L = U\Psi_L$ can be made, in which case neither the diagonal μ_{RL} nor $M_{LL} = 0$ appears to change. The (12×12) matrix that results is:

$$U^T M U = \begin{vmatrix} M_{R1} & 0 & \mu & 0 \\ & \ddots & & \ddots \\ 0 & M_{R6} & 0 & \mu \\ \mu & 0 & 0 & 0 \\ & \ddots & & \ddots \\ 0 & \mu & 0 & 0 \end{vmatrix}. \quad (31)$$

This matrix splits into a product of twofold matrices we are well familiar with from works on the see-saw mechanism (see review [3]):

$$m^D = \begin{vmatrix} M_D & \mu \\ \mu & 0 \end{vmatrix}, \quad D = 1, 2, \dots, 6. \quad (32)$$

At $\mu \ll M_D$, two eigenvalues of m^D differ from each other in magnitude by a factor of m^2/M_D^2 :

$$\lambda_{1,2}^D = \frac{M_D}{2} \pm \sqrt{\frac{M_D^2}{4} + \mu^2} \simeq \begin{cases} M_D + \frac{\mu^2}{M_D} \\ -\frac{\mu^2}{M_D} \end{cases}. \quad (33)$$

Formula (33) is valid for any sign of M_D . Note that this sharp distinction λ may result only if the second element on the principal diagonal in (32) is equal to zero, i.e., at the corresponding mass $M_L^{(D)} = 0$. This is the physical meaning of the condition $M_{LL} = 0$ in Section 5. For variant (a) in Section 5, $M_{LL} \neq 0$ and even $M_R = M_L$, the spectrum of two neutrinos ($n = 2$) will have an absolutely different form.

Eigenfunctions (32) are determined by rotation of the matrix orths through the transformation:

$$V_D = \begin{vmatrix} \alpha_D & \beta_D \\ -\beta_D & \alpha_D \end{vmatrix}, \quad (34)$$

where quantities α_D and β_D are equal to:

$$\alpha_D = \frac{1}{\sqrt{1 + (\mu/M_D)^2}}, \quad \beta_D = \frac{\mu/M_D}{\sqrt{1 + (\mu/M_D)^2}}, \quad \alpha^2 + \beta^2 = 1. \quad (35)$$

The signs in Eq. (35) are chosen for further convenience.

In matrix (31), let us place masses M_{Ri} so that $M_{R6} = -M_{R1}$, $M_{R5} = -M_{R2}$, $M_{R4} = -M_{R3}$. Then, for mass matrix (18), we have three pairs of heavy masses: $M_{\pm D}$, $D = 1, 2, 3$, and three pairs of light masses, $m_{\pm D}$, ($\pm D$ means the sign of the mass D). The eigenfunctions of diagonalized states are:

$$\Psi_{\pm D} = U_{\pm D}^a (\alpha_{\pm D} \Psi_{Ra} + \beta_{\pm D} \Psi_{La}), \quad (36)$$

$$\psi_{\pm D} = U_{\pm D}^a (-\beta_{\pm D} \Psi_{Ra} + \alpha_{\pm D} \Psi_{La}).$$

Depending on the sign selected (35), we have:

$$\alpha_D = \alpha_{-D}, \quad \beta_D = -\beta_{-D}. \quad (37)$$

Wavefunctions (36) have a property similar to Majorana relations (11), (12). Using these relations, one can establish a connection between states (36) with different mass signs:

$$\gamma_5 h C \bar{\Psi}^{TD} = \Psi_{-D}, \quad (38)$$

$$\gamma_5 h C \bar{\psi}^{TD} = -\psi_{-D}.$$

Mass disposition is chosen so that $h_{D'D} \equiv h_{-DD}$. We, therefore, can continue using covariant formulae, raising and lowering the diagonal indices D . In order to prove (38), we first write:

$$\gamma_5 h C \bar{\Psi}^{TD} = h U^{+T} h^+ \left(\alpha_D \gamma_5 h C \bar{\Psi}_R^T + \beta_D \gamma_5 h C \bar{\Psi}_L^T \right). \quad (39)$$

From formula (13) written in the covariant form, it follows that there is a relation between the elements of matrices U that diagonalize M_{RR} :

$$h_{D'D} U^{+TD}{}_b h^{ba} = \pm U_{-D}{}^a; \quad D' = -D; \quad a, b = 1, 2, \dots, 6. \quad (40)$$

Taking into consideration (11), (12) and sign change for β with the sign change for the mass M_D , we finally obtain (38).

Equations (38) facilitate construction of physical Dirac states with positive masses. Let us first construct true Majorana states ($\psi = C \bar{\psi}^T$) for each of the diagonal states $D = 1, 2, 3$. We have

$$\begin{aligned} \chi_1^{(M_D)} &= \frac{\Psi_D + C \bar{\Psi}^{TD}}{\sqrt{2}} = \frac{\Psi_D + \gamma_5 h^+ \Psi_{-D}}{\sqrt{2}}, \\ \chi_2^{(M_D)} &= \frac{\Psi_D - C \bar{\Psi}^{TD}}{\sqrt{2}i} = \frac{\Psi_D - \gamma_5 h^+ \Psi_{-D}}{\sqrt{2}i}, \end{aligned} \quad (41)$$

which will be similar also for states with light masses $\chi_1^{(m_D)}$ and $\chi_2^{(m_D)}$.

Massive Dirac states with positive masses are expressed as follows:

$$\Psi = \frac{\chi_1 + i\chi_2}{\sqrt{2}}, \quad \bar{\Psi} = \frac{\bar{\chi}_1^T - i\bar{\chi}_2^T}{\sqrt{2}} C^{+T}, \quad (42)$$

for any $M_D > 0$, $m_D > 0$.

It is well known that transformations (41) and (42) transfer the free Lagrangians for Majorana particles into the Lagrangian for Dirac particles. Thus, twelve Majorana states will contribute to three heavy and three light Dirac particles.

VII. MASS MATRIX M SPECTRUM AND LIGHT NEUTRINO MASSES

Let us evaluate what kind of spectra can be obtained from matrix (18) in the scheme under consideration at $n = 6$, $M_{LL} = 0$, and the diagonal Dirac form μ_{RL} . The Majorana mass matrix M_{RR} obeys conditions (13). Let us assume that scales $\mu \ll M$, in order to have the states split into light and heavy masses (the see-saw mechanism). Then, M_{RR} can be

written as:

$$M_{RR} = M \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{12} & a_{22} & a_{23} & a_{24} & a_{25} & -a_{15} \\ a_{13} & a_{23} & a_{33} & a_{34} & -a_{24} & a_{14} \\ a_{14} & a_{24} & a_{34} & -a_{33} & a_{23} & -a_{13} \\ a_{15} & a_{25} & -a_{24} & a_{23} & -a_{22} & a_{12} \\ a_{16} & -a_{15} & a_{14} & -a_{13} & a_{12} & -a_{11} \end{vmatrix}. \quad (43)$$

We have twelve independent elements that take certain values imposed by the *gap* equations. Since the equations are not known to us, we have to limit ourselves to a rough estimate of values resulting from the matrix form (43).

Let us take all independent elements $M_{RR}/M \simeq 1$. There is no reason to think that the equations with high symmetry will result in parameters being essentially different from each other. For these parameters we cannot imagine any other physical justified distribution pattern .

The only difference inherent in (43) is the difference in signs: ± 1 (in M units), so one can take independent elements with opposite signs. We have checked several variants, all of which lead to eigenvalue equations with coefficients alternating in sign. The second coefficient of the equation is obviously negative (at $(M_D^2)^2$).

If all independent elements $a_{ik} = 1$, then the eigenvalue equation M_{RR} is written as ($x = M_D^2/M^2$):

$$x^3 - 18x^2 + 48x - 32 = 0, \quad (44)$$

with roots $x_1 \simeq 1.02$, $x_2 = 2$, and $x_3 \simeq 14.92$. Light masses squared (in $(\mu^2/M)^2$ units) are:

$$m_1^2 \simeq 0.067, \quad m_2^2 = 0.5, \quad m_3^2 \simeq 1.$$

Experimentally obtained ratio [1], Eq. (1)

$$\left| \frac{\Delta m_{23}^2}{\Delta m_{12}^2} \right| = \frac{m_3^2 - m_2^2}{m_2^2 - m_1^2} \simeq 3, \quad (45)$$

is, of course, far from the experimental value in Section 1.

Other variants of $a_{ik} = \pm 1$ lead to various equations. The eigenvalue equation

$$x^3 - 18x + 80x - 64 = 0$$

results if the matrix elements are negative: 1) $a_{13} = -1$; 2) $a_{14} = -1$; 3) $a_{15} = -1$; and so on. Ratio (45) here is equal to ~ 10 . The equation

$$x^3 - 18x^2 + 64x - 32 = 0$$

can be obtained at 1) $a_{22} = -1$; 2) $a_{22} = -1$, $a_{16} = -1$; and so on. Ratio (45) here is equal to ~ 8 .

One can obtain equations with one real root, which is physically unacceptable:

$$x^3 - 12x^2 + 96x - D = 0$$

$D = 32$ for $a_{14} = a_{15} = a_{13} = -1$; $D = 126$ at $a_{12} = -1$; $D = 136$ at $a_{14} = a_{15} = a_{13} = a_{16} = -1$.

Finally, there is a frequent situation when the least root x appears to be smaller than 1, whereas the other two roots are larger than 1. Then, ratio (45) is big, which is required phenomenologically (see [1]). The equation

$$x^3 - 18x^2 + 80x - D = 0 \tag{46}$$

is valid for $D = 32$, $a_{12} = a_{15} = -1$, roots $x_1 \simeq 0.045$, $x_2 \simeq 6.51$, $x_3 \simeq 11.04$ and ratio (45) equal to $\simeq 35$; for $D = 24$, $a_{23} = a_{15} = -1$, roots $x_1 \simeq 0.325$, $x_2 \simeq 6.8$, $x_3 \simeq 10.9$, ratio (45) equal to $\simeq 53$. The coefficient values being high, the small quantity x_1 coexists with the rather large x_2 and x_3 , facilitating fitting big ratios (45).

We note that big numbers (45) are relatively easy to obtain in matrices of the type under consideration. The character of the spectrum obtained experimentally, where one neutrino state is located far from the other, almost degenerate ones, means, in terms of the present investigation, that $x_1 < 1$, while the other two $x_{2,3} > 1$. This kind of situation is not rare in matrices of the (43) type and there is nothing unique to it.

The light particle mass spectrum shows normal hierarchy: the heavy mass corresponds to the large difference Δm_{23}^2 . The small difference Δm_{12}^2 is achieved with the Majorana masses $x_{2,3} > 1$. At that, no degeneration of states 1,2 occurs; what is only noted is the effect of difference for inverse squares of large numbers. In terms of the Majorana spectrum, the situation is quite ordinary.

VIII. CONCLUSION

The scheme under consideration, if ever possible [8, 9], does not include many aspects of the phenomena that are essential under a less simplified model (renormalization of non-perturbative solutions, scale dependence of various factors). On the other hand, it has a number of attractive features. Of interest is the connection in (25) of the light neutrino number and spectrum with the compulsory spontaneous breaking of the $-R, L$ -symmetry. One needs to gain deeper insight into the meaning of this connection.

Spectrum reproduction indicates that the reason for the specific neutrino mass disposition is rather banal: the smallness of one of the Majorana masses (using comparative units) $M_1^2 < 1$ and the relatively heavy two others $M_{2,3}^2 > 1$. This scheme results in normal mass hierarchy of neutrino states.

Mixing angles for observed neutrino flavours can not be considered unless the mechanism of mass spectrum generation is determined for charged leptons. The next step, therefore, is to include charged, and, as is well known, exclusively Dirac leptons in the investigation of weak interactions. In this connection, the R, L -symmetry violation, which occurs spontaneously, appears to be significant as it means symmetry violation outside the Standard Model.

This phenomenon not taken into account, one has to overcome the following difficulty. The Standard left weak current constructed by means of chiral states ψ_L , when written in terms of Majorana particles and then in terms of massive Dirac particles, appears to also include their right- hand components (or antiparticles):

$$\bar{\psi}_L^a \gamma_\mu \psi_{La} \equiv \frac{1}{2} \bar{\Psi}_L^a \gamma_\mu \gamma_5 \Psi_{La}, \quad (47)$$

since $\bar{\Psi}_L^a \gamma_\mu \Psi_{La} \equiv 0$ (see section 4). Including only left components of physical Dirac states ψ_D, Ψ_D , in the weak current, which is attractive from the phenomenological point of view, would break the $Sp(n/2)$ -symmetry once again. This would mean that the joint ($Sp(n/2)$) and weak Standard models are non-renormalizable.

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Appendix 1. Mass of Gauge Boson F_μ

In order the vector boson F_μ^A , ($A = 1, 2, \dots, n(n+1)/2$) acquires a mass, it is sufficient that the scalar field is present under the same adjoint representation as F_μ^A . The scalar field ϕ^A is expressed by means of the symmetric tensor ϕ_{ab} (or $\phi^{ab} = h^{aa'}h^{bb'}\phi_{a'b'}$):

$$\phi^A = t_a^{A\ b} h_{bb'} \phi^{ab'} = \phi_{ab'} h^{b'b} t_b^{A\ a} \equiv t_a^{A\ b} \phi_b^{\ a}. \quad (\text{A.1})$$

All of these tensors have $N = n(n+1)/2$ independent components, $n(n+1)$ real parameters.

The Standard Higgs Lagrangian, which is $Sp(n/2)$ - invariant

$$\phi_a^{+b} \left(\partial_\mu \delta_b^c + i t_b^{Ac} F_\mu^A(x) \right) \left(\partial_\mu \delta_c^d - i t_c^{A'd} F_\mu^{A'} \right) \phi_d^{\ a}(x) - V(|\phi|), \quad (\text{A.2})$$

with simple $V(|\phi|)$ chosen so that $|\phi|^2 = \phi_a^{+b} \phi_b^{\ a}$ acquires the vacuum average

$$\langle \phi_a^{+b} \phi_b^{\ a'} \rangle = \frac{1}{N} \eta^2 \delta_a^{\ a'}, \quad (\text{A.3})$$

creates masses for all F_μ^A . At that, all phases of ϕ are absorbed in the formation of heavy masses for F_μ^A , so that ϕ^A are present as really existing particles only at very high energies. Large number of possible $Sp(n/2)$ invariant forms permits to use, besides (A.3), also more complicated spectra of heavy ϕ particles.

Appendix 2.

Hermitian matrices t^A in the representation n of the $Sp(n/2)$ group can be selected from $n^2 - 1$ infinitesimal operators T_A in the fundamental representation $N = n$ of the better known group $SU(N)$. The even-dimensional operators T_A can be constructed to form two groups with respect to operation (2) with the skew antisymmetrical matrix $h(n \times n)$, Eq. (14):

$$h^+ t^A h = -t^{AT}, \quad h^+ \sigma^P h = \sigma^{PT}. \quad (\text{A.4})$$

We have $n(n+1)/2$ matrices t^A and $[\frac{1}{2}n(n-1) - 1]$ matrices σ^P [4]. The operators t^A , σ^P and unity operator I form the complete basis for space of $(n \times n)$ matrices, with its usual normalization being:

$$Tr t^A t^B = \frac{1}{2} \delta^{AB}, \quad Tr \sigma^P \sigma^Q = \frac{1}{2} \delta^{PQ}, \quad Tr t^A \sigma^P = 0. \quad (\text{A.5})$$

Expanding the right parts of commutators and anticommutators t and σ in the complete basis and using Eqs. (A.4), we obtain:

$$\begin{aligned} [t^A, t^B] &= i f^{ABC} t^C, & \{t^A, t^B\} &= \frac{1}{2n} \delta^{AB} I + d^{ABP} \sigma^P, \\ [t^A, \sigma^P] &= i f^{APQ} \sigma^Q, & & \\ [\sigma^P, \sigma^Q] &= i f^{PQA} t^A, & \{\sigma^P, \sigma^Q\} &= \frac{1}{2n} \sigma^{PQ} I + d^{PQR} \sigma^R. \end{aligned} \quad (\text{A.6})$$

The coefficients d^{PQR} in the anticommutator σ are equal to zero, as it possible to show. From Eq.(A.6) it is evident that the matrices t^A form a closed algebra, which in turn is responsible for the representation n of the group $Sp(n/2)$. Matrices σ^P are responsible for another irreducible representation of this group with dimension $[\frac{1}{2}n(n-1) - 1]$.

Let us use the well-known equation for the infinitesimal operators in the fundamental representation $SU(n)$. In terms of the operators t^A and σ^P , we have (summation over indices A and P):

$$t_a^A t_c^A t^d + \sigma_a^{Pb} \sigma_c^{P d} = \frac{1}{2} \left(\delta_a^d \delta_c^b - \frac{1}{n} \delta_a^b \delta_c^d \right). \quad (\text{A.7})$$

After applying Eq.(A.4) to both parts of (A.7), changing the indices, adding and subtracting (A.7) and the equation obtained, we have:

$$t_a^A t_c^A t^d = \frac{1}{4} \left(\delta_a^d \delta_c^b - h^{bd} h_{ca} \right) \quad (\text{A.8})$$

$$\sigma_a^{Pb} \sigma_c^{P d} = \frac{1}{4} \left(\delta_a^d \delta_c^b + h^{bd} h_{ca} - \frac{2}{n} \delta_a^b \delta_c^d \right). \quad (\text{A.9})$$

With the usual notations, $(t^{AT})^a_b = t_b^{Aa}$, which is used in the calculations.

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